Predicates and Quantifiers

Propositional logic cannot adequately express the meaning of all statements in mathematics and in natural language.

Example-1

“Every computer connected to the university network is functioning properly”

“MATH3 is one of the computers connected to the university network”.

No rules of propositional logic allow us to conclude the truth of the statement “MATH3 is functioning properly,”

Example-2

* + “CS2 is under attack by an intruder,”
  + “CS2 is a computer on the university network”,
* No rules of propositional logic allow us to conclude the truth of the statement
  + “There is a computer on the university network that is under attack by an intruder.”

It generates the concept of Predicate Logic.

Predicate is the part of a sentence or clause containing a verb and stating something about the subject.

The statement “x is greater than 3” has two parts. The first part, the variable x, is the subject of the statement. The second part, the **predicate**, “is greater than 3. We can denote the statement “x is greater than 3” by P(x), where P denotes the predicate “is greater than 3” and x is the variable. The statement P(x) is also said to be the value of the **propositional function** P at x. Once a value has been assigned to the variable x, the statement P(x) becomes a proposition and has a truth value.

**Example:** Let A(x) denotes the statement “Computer x is under attack by an intruder.”

Suppose that, from the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of *A*(CS1), *A*(CS2), and *A*(MATH1)?

**Solution:** We obtain the statement *A*(CS1) by setting x = CS1 in the statement “Computer *x* is under attack by an intruder.” Because CS1 is not on the list of computers currently under attack, we conclude that *A*(CS1) is false. Similarly, because CS2 and MATH1 are on the list of computers under attack, we say that *A*(CS2) and *A*(MATH1) are true.

**Example:** Let *Q(x, y)* denote the statement “*x* = *y* + 3.” What are the truth values of the propositions Q*(*1*,* 2*)* and *Q(*3*,* 0*)*?

**Solution:** To obtain *Q(*1*,* 2*)*, set *x* = 1 and *y* = 2 in the statement *Q(x, y)*. Hence, *Q(*1*,* 2*)* is the statement “1 = 2 + 3,” which is false. The statement *Q(*3*,* 0*)* is the proposition “3 = 0 + 3,” which is true.

**Quantifier:** It quantifies the subject of the statement i.e., it defines the range in which the predicate is true. Basically two types of quantifiers are there, Universal Quantifier and Existential Quantifier.

The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

The Universal Quantifier**:** Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the **domain of discourse** (or the **universe of discourse**), often just referred to as the **domain**. Such a statement is expressed using universal quantification.

The **universal quantification** of P(x) is the statement

“P(x) for all values of x in the domain.”

The notation ∀x P(x) denotes the universal quantification of P(x). Here ∀ is called the **universal quantifier.** We read ∀x P(x) as “for all x, P(x)” or “for every x , P(x).” An element for which P(x) is false is called a **counterexample** of ∀x P(x).

**Example:** Let P(x) be the statement *“*x + 1 > x.*”* What is the truth value of the quantification ∀x P(x), where the domain consists of all real numbers ?

**Solution:** Because P(x) is true for all real numbers x, the quantification ∀x P(x) is true.

**Example:** Let P(x) be the statement “x < 2.” What is the truth value of the quantification

∀x P(x), where the domain consists of all real numbers?

**Solution***:* P(x) is not true for every real number x, because, for instance, P(3) is false. That is,

x = 3 is a counterexample for the statement ∀x P(x). Thus ∀x P(x) is false.

**Example:** What is the truth value of ∀x P(x), where P(x) is the statement *“*x2 < 10” and the domain consists of the positive integers not exceeding 4 ?

**Solution***:* The statement ∀x P(x) is same as the conjunction P(1) ∧ P(2) ∧ P(3) ∧P(4), because the domain consists of the integers 1, 2, 3, and 4. Because P(4), which is the statement *“*42< 10, *”* is false, it follows that ∀x P(x) is false.

**Example:** Express the statement “Every student in this class has studied calculus” using predicates and quantifiers.

**Solution:** First, we rewrite the statement so that we can clearly identify the appropriate quantifiers to use.

“For every student in this class, that student has studied calculus.”

Next, we introduce a variable x so that our statement becomes

“For every student x in this class, x has studied calculus.”

Again, we introduce P(x), which is the statement “x has studied calculus”. Consequently, if the

domain for x consists of the students in the class, we can translate our statement as ∀x P(x).

**Note:** Besides “for all” and “for every,” universal quantification can be expressed in many other ways, including “all of,” “for each,” “given any,” “for arbitrary,” “for each,” and “for any.” It is best to avoid using “for any *x*” because it is often ambiguous as to whether “any” means “every” or “some.” In some cases, “any” is unambiguous, such as when it is used in negatives: “There is not any reason to avoid studying.”

The Existential Quantifier: Many mathematical statements assert that there is an element with a certain property, such statements are expressed using existential quantification.

In existential quantification, we form a proposition that is true if and only if P(x) is true for at least one value of x in the domain.

The **existential quantification** of P(x) is the proposition

“There exists an element x in the domain such that P(x) .”

We use the notation ∃x P(x) for the existential quantification of P(x). Here ∃ is called the

# existential quantifier.

**Note*:*** Besides the phrase “there exists,” we can also express existential quantification in many other ways, such as by using the words “for some,” “for at least one,” or “there is.”

The existential quantification ∃x P(x) is read as “There is an x such that P(x) ,” “There is at least one *x* such that P(x)”, “For some x P(x)”.

Observe that the statement ∃x P(x) is false if and only if there is no element x in the domain for which P(x) is true. That is, ∃x P(x) is false if and only if P(x) is false for every element of the domain.

**Example:** Let P(x) denote the statement “x > 3.” What is the truth value of the quantification ∃x P(x), where the domain consists of all real numbers?

**Solution:** Because “x > 3” is sometimes true, for instance when *x* = 4, the existential quantification of P(x), which is ∃x P(x), is true.

**EXAMPLE:** Let P(x) denote the statement “x = x + 1.”

What is the truth value of the quantification ∃x P(x), where the domain consists of all real numbers ?

**Solution:** Because P(x) is false for every real number x, the existential quantification of P(x),

i.e., ∃x P(x*)*, is false.

**Example:** Express the statements “Some student in this class has visited Mexico” using predicates and quantifiers.

**Solution:** The statement “Some student in this class has visited Mexico” means that “There is a student in this class with the property that the student has visited Mexico.”

We can introduce a variable x, so that our statement becomes

“There is a student x in this class having the property that x has visited Mexico.”

We introduce P(x), which is the statement “x has visited Mexico.” If the domain for x consists of the students in this class, we can translate this first statement as ∃x P(x).

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| **Quantifiers.** | | |
| ***Statement*** | ***When True?*** | ***When False?*** |
| ∀x P(x) | P(x) *is true for every* x*.* | There is an x for which P(x) is false. |
| ∃x P(x) | *There is an* x *for which* P(x) *is true.* | P(x) is false for every x. |

Quantifiers Over Finite Domains

When the domain of a quantifier is finite, that is, when all its elements can be listed, quantified

statements can be expressed using propositional logic. In particular, when the elements of the domain are x1, x2, … , xn, where n is a positive integer, the universal quantification ∀x P(x) is the same as the conjunction

P(x1) ∧ P(x2) ∧ ⋯ ∧ P(xn),

because this conjunction is true if and only if P(x1), P(x2), ⋯ , P(xn) are all true. Similarly the existential quantification ∃x P(x) is the same as the conjunction

P(x1) ∨ P(x2) ∨ ⋯ ∨ P(xn).

Quantifiers with Restricted Domains: Consider the statement “For every natural number x,

x2 < 100.” In this case the domain of discourse is the set of all natural numbers {1,2,3, … }. The given statement is ∀x P(x) , where P(x) is the statement “x2 < 100” and it is false as P(11) is false. But if we consider our domain of discourse to be the set A = {1,2,3,4,5,6,7,8,9}, then x2 < 100 is true. So if we change the domain of a quantifier the truth value of proposition may be changed.

Let Q(x) be the proposition “ x is in A”. Then the proposition “For every x in A, x2 < 100” is written as ∀x (Q(x) → P(x)), where the domain of discourse may be the set of all natural numbers or real number. Another way of writing the statement ∀x (Q(x) → P(x)) is ∀x ∈ A, P(x).

The restricted domain statement ∃x ∈ A P(x) reads, "There exists some x in A, , x2 < 100". This is equivalent to saying, "There exists some x, that is in A and the predicate P(x) holds", which is ∃x (x ∈ A ∧ P(x)).

**Example:** What do the statements ∀x < 0 (x2 > 0), ∀y ≠ 0 (y3 ≠ 0), and ∃z > 0 (z2 =

2) mean, where the domain in each case consists of the real numbers?

**Solution:** The statement ∀x < 0 (x2 > 0), states that for every real number x with

x < 0, x2 > 0. That is, it states “The square of a negative real number is positive.” This statement is the same as ∀x (x < 0 → x2 > 0). The statement ∀y ≠ 0 (y3 ≠ 0), states that for every real number y with y ≠ 0, we have y3 ≠ 0. That is, it states “The cube of every nonzero real number is nonzero.” Note that this statement is equivalent to ∀y (y≠ 0 →y3 ≠ 0). Finally, the statement ∃ z > 0 (z2 = 2) states that there exists a real number *z* with z > 0 such that z2 = 2. That is, it states “There is a positive square root of 2.” This statement is equivalent to ∃z (z > 0 ∧ z2 = 2).

Logical Equivalences Involving Quantifiers: Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.

**Example:** Show that ∀x (P(x) ∧ Q(x)) and ∀x P(x) ∧ ∀x Q(x) are logically equivalent. **Solution:** To show that these statements are logically equivalent, we must show that they always take the same truth value, no matter what the predicates *P* and *Q* are, and no matter which domain of discourse is used. Suppose we have particular predicates *P* and *Q*, with a common domain. We can show that ∀x (P(x) ∧ Q(x)) and ∀x P(x) ∧ ∀x Q(x) are logically equivalent by doing two things. First, we show that if ∀x (P(x) ∧ Q(x)) is true, then ∀x P(x) ∧ ∀x Q(x) is true. Second, we show that if ∀x P(x) ∧ ∀x Q(x) is true, then ∀x (P(x) ∧ Q(x)) is true.

So, suppose that ∀x P(x) ∧ ∀x Q(x) is true. This means that if *a* is in the domain, then P(a) ∧ Q(a) is true. Hence, *P(a)* is true and *Q(a)* is true. Because *P(a)* is true and *Q(a)* is true for every element in the domain, we can conclude that ∀x P(x) and ∀x Q(x) are both true. This means that ∀x P(x) ∧ ∀x Q(x) is true.

Next, suppose that ∀x P(x) ∧ ∀x Q(x) is true. It follows that ∀x P(x) is true and ∀x Q(x) is true. Hence, if *a* is in the domain, then *P(a)* is true and *Q(a)* is true [because *P(x)* and *Q(x)* are both true for all elements in the domain, there is no conflict using the same value of *a* here]. It follows that for all *a*, P(a) ∧Q(a) is true. It follows that ∀a (P(a) ∧ Q(a)) is true. We can now conclude that ∀x (P(x) ∧ Q(x)) ≡ ∀x P(x) ∧ ∀x Q(x).

**Note:** This logical equivalence shows that we can distribute a universal quantifier over a conjunction. Furthermore, we can also distribute an existential quantifier over a disjunction.

However, we cannot distribute a universal quantifier over a disjunction, nor can we distribute an existential quantifier over a conjunction.

**Negating Quantified Expressions**

We will often want to consider the negation of a quantified expression. For instance, consider the negation of the statement

“Every student in your class has taken a course in calculus.” This statement is a universal quantification, namely,

∀x P(x),

where P(x) is the statement “x has taken a course in calculus” and the domain consists of the students in your class. The negation of this statement is “It is not the case that every student in your class has taken a course in calculus.” This is equivalent to “There is a student in your class who has not taken a course in calculus.” And this is simply the existential quantification of the negation of the original propositional function, namely,

∃x ~P(x).

This example illustrates the following logical equivalence: **~(∀x P(x)) ≡ ∃x (~P(x))**

The rules for negations for quantifiers are called **De Morgan’s laws for quantifiers.**

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| Negations | Equivalent  statement | When is Negation true? | When is Negation false? |
| ~∃x P(x) | ∀x ~P(x) | For every x, P(x) is false. | There is an x for which P(x)  is true |
| ~∀x P(𝑥) | ∃x ~P(𝑥) | There is an x for which P(x)  is false | For every x, P(x) is true. |

**Example**:What are the negations of the statements “There is an honest politician” and “All Americans eat cheeseburgers” ?

**Solution:** Let H(x) denote “x is honest.” Then the statement “There is an honest politician” is

represented by ∃x H(x), where the domain consists of all politicians. The negation of this statement is ~∃x H(x), which is equivalent to ∀x ~H(x). This negation can be expressed as “Every politician is dishonest.”

Let C(x) denote “x eats cheeseburgers.” Then the statement “All Americans eat cheeseburgers” is represented by ∀x C(x), where the domain consists of all Americans. The negation of this statement is ~∀x C(x), which is equivalent to ∃x ~C(x). This negation can be

expressed in several different ways, including “Some Americans do not eat cheeseburgers” and “There is an American who does not eat cheeseburgers.”

Using Quantifiers in System Specifications: Previously, we used propositions to represent system specifications. However, many system specifications involve predicates and quantifications.

**Example:** Use predicates and quantifiers to express the system specifications “All lions are fierce.”

“Some lions do not drink coffee.”

“Some fierce creatures do not drink coffee.”

**Solution***:* The first two statements are called *premises* and the third is called the *conclusion*. The entire set is called an *argument*.

Let P(x), Q(x), and R(x) be the statements “x is a lion,” “x is fierce,” and “x drinks coffee,”

respectively. We assume that the domain consists of all creatures. We can express these statements as:

∀x (P(x) → Q(x)).

∃x (P(x) ∧ ~R(x)).

∃x (Q(x) ∧ ~R(x)).

Notice that the second statement cannot be written as ∃x (P(x) → ~R(x))and the third statement cannot be written as ∃x (Q(x) → ~R(x)).

Exercises

1. Let *P*(x) be the statement “the word x contains the letter a.” What are these truth values?
   1. *P*(orange) **b)** *P*(lemon)

**c)** *P*(true) **d)** *P*(false)

1. Let P(x) be the statement “x = x2.” If the domain consists of the integers, what are these truth values?

**a)** *P(*0*)* **b)** *P(*1*)* **c)** *P(*2*)* **d)** *P(*−1*)* **e)** ∃x P(x) **f )** ∀x P(x)

1. Determine the truth value of each of these statements if the domain consists of all integers.

**a)** ∀n (n + 1 > n) **b)** ∃n (2n = 3n) **c)** ∃ n (n = −n) **d)** ∀n (3n ≤ 4n)

1. Determine the truth value of each of these statements if the domain consists of all real numbers.

**a)** ∃x (x3 = −1) **b)** ∃x (x4 < x2) **c)** ∀x ((−x)2 = x2) **d)** ∀x (2x > x)

1. Suppose that the domain of the propositional function *P(x)* consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.

**a)** ∃x P(x) **b)** ∀x P(x) **c)** ∃x ~P(x) **d)** ∀x ~P(x) **e)** ~∃x P(x) **f )** ~∀x P(x)

Rules of Inference for Quantified Statements: We have discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

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| **Rules of Inference for Quantified Statements.** | |
| ***Rule of Inference*** | ***Name*** |
| ∀x P(x)  ∴ P(c) for an arbitrary *c* | Universal instantiation |
| P(c) for an arbitrary *c*  ∴ ∀x P(x) | Universal generalization |
| ∃x P(x)  ∴ P(c) for some element c | Existential instantiation |
| P(c) for some element *c*  ∴ ∃x 𝑃(x) | Existential generalization |

**Example:** Show that the premises “Everyone in this discrete mathematics class has taken a course in computer science” and “Marla is a student in this class” imply the conclusion “Marla has taken a course in computer science.”

**Solution:** Let *D(x)* denote “*x* is in this discrete mathematics class,” and let *C(x)* denote “*x* has taken a course in computer science.” Then the premises are ∀x (D(x) → C(x)) and *D*(Marla). The conclusion is *C*(Marla).

The following steps can be used to establish the conclusion from the premises.

# Step Reason

1. ∀x (D(x) → C(x)) Premise

1. *D*(Marla)→*C*(Marla) Universal instantiation from (1)
2. *D*(Marla) Premise
3. *C*(Marla) Modus ponens from (2) and (3)

**Example:** Show that the premises “A student in this class has not read the book,” and “Everyone in this class passed the first exam” imply the conclusion “Someone who passed the first exam has not read the book.”

**Solution***:* Let *C(x)* be “*x* is in this class,” *B(x)* be “*x* has read the book,” and *P(x)* be “*x* passed the first exam.”

The premises are ∃x (C(x) ∧ ~B(x)) and ∀x (C(x) → P(x)). The conclusion is

∃x (P(x) ∧ ~B(x)). These steps can be used to establish the conclusion from the premises.

# Step Reason

1. ∃x (C(x) ∧ ~B(x)) Premise

2. C(a) ∧ ~B(a)) Existential instantiation from (1)

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| 3. *C(a)*  4. ∀ (C(x) | → P(x)) | | Simplification from (2)  Premise | |
| 5. C(a) → | P(a) | Universal instantiation from (4) | |
| 6. *P(a)*  7. ~B(a) |  | Modus ponens from (3) and (5)  Simplification from (2) | |

8. P(a) ∧ ~B(a) Conjunction from (6) and (7)

1. ∃x (P(x) ∧ ~B(x)) Existential generalization from (8)

Combining Rules of Inference for Propositions and Quantified Statements: We have developed rules of inference both for propositions and for quantified statements. Sometimes we use both a rule of inference for quantified statements, and a rule of inference for propositional logic.

Here we give some of them

* 1. **Universal modus ponens** rule tells us that if ∀x (P(x) → Q(x)) is true, and if *P(a)* is true for a particular element *a* in the domain of the universal quantifier, then *Q(a)*

must also be true.

* 1. **Universal modus tollens** combines universal instantiation and modus tollens and can be expressed in the following way:

∀x P(x) → Q(x))

~Q(a)*,* where a is a particular element in the domain

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∴ ~P(a)

Exercises

1. For each of these arguments, explain which rules of inference are used for each step.
   1. “Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution.”
   2. “All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coalminers.”
2. For each of these arguments determine whether the argument is correct or incorrect and explain why.
   1. All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.
   2. Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.
3. For each of these collections of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.
   1. “Every computer science major has a personal computer.” “Ralph does not have a personal computer.” “Ann has a personal computer.”
   2. “All rodents gnaw their food.” “Mice are rodents.”“Rabbits do not gnaw their food.” “Bats are not rodents.”